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Oscillations of Fourier Coefficients of Cusp Forms over Primes

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1. INTRODUCTION AND MAIN RESULTS

The aim of this survey is to introduce our recent work [7], [8] on oscillations of Fourier coefficients of cusp forms over primes twisted with additive functions.

Sums concerning prime numbers are important problems in analytic number theory. Hence, number theorists are interested in sums of type

$$\sum_{n\leq x}\Lambda(n)a_ne(n^k\theta)\quad \text{and}\quad \sum_{n\leq x}\mu(n)a_ne(n^k\theta), \tag{1.1}$$

where $\mathcal{A}=(a_n)$ is an arithmetic sequence of complex numbers. Here, as usual, $\Lambda(n)$ and $\mu(n)$ are the von Mangoldt function and the Möbius function.

The problem that $a_n\equiv 1$ was first considered by Davenport [2] for $k=1$. He showed, by Vinogradov's ingenious method on estimating exponential sums over primes, that

$$\sum_{n\leq x}\mu(n)e(n\theta)\ll_A \frac{x}{\log^A x} \tag{1.2}$$

for any $A\geq 0$, where the implied constant depends only on A. For the nonlinear case $k\geq 2$, by applying the techniques introduced by Hua [4, Theorem 10], one can show that

$$\sum_{n\leq x}\mu(n)e(n^k\theta)\ll_{A,k} \frac{x}{\log^A x} \tag{1.3}$$

for any $A\geq 0$, where the implied constant depends only on A and k.

Following Fouvry and Ganguly [3], two sequences (x_n) and (y_n) of complex numbers are strong asymptotically orthogonal if

$$\sum_{1\leq n\leq N}x_ny_n=O_A\left((\log N)^{-A}\sum_{n\leq N}|x_ny_n|\right)$$

for every $A\geq 0$, uniformly for $N\geq 2$. Hence Davenport's result means that uniformly for all real numbers α, the sequences $\{\mu(n)\}$ and $\{e(n^k\alpha)\}$ are strong asymptotically orthogonal.

Recently, when $k=1$ and $a_n=a_F(n)$, where $a_F(n)$ are the normalized Fourier coefficients of holomorphic or Maass cusp forms F for $SL(2,\mathbb{Z})$, Fouvry and Ganguly [3] proved that

$$\sum_{n\leq x}\Lambda(n)a_F(n)e(n\theta)\ll_{F,c} x\exp(-c\sqrt{\log x}) \tag{1.4}$$

for some constant $c>0$. In fact, this is a classical problem, see e.g. Perelli [10]. They also derived the corresponding result

$$\sum_{n\leq x}\mu(n)a_F(n)e(n\theta)\ll_{F,c} x\exp(-c\sqrt{\log x}). \tag{1.5}$$

This can be regarded as one GL(2) analogue of Davenport's theorem. It is also closely related to the the Möbius randomness law, which recently appeals to many authors.

Recently, Hou and Lü [7] and Hou, Jiang and Lü [8] were able to generalize the result of Fouvry and Ganguly to the following cases.

Theorem 1.1. *Let $N\geq 2$ and F be a primitive holomorphic or Maass cusp form for the group $SL(2,\mathbb{Z})$. Let $a_F(n)$ denote the nth normalized Fourier coefficient of the form F. Then there exists an effective absolute $c_1>0$ such that, for any $\alpha\in\mathbb{R}$ and any quadratic polynomial $g(n)$ with integral coefficients,*

$$\sum_{n\leq N}\Lambda(n)a_F(n)e(g(n)\alpha)\ll N\exp(-c_1\sqrt{\log N}).$$

where the implied constant depends only on the form F.

Theorem 1.2. *Let $N\geq 2$ and $L(s,F)$ be the L-function associated to a Hecke-Maass form F for $SL(3,\mathbb{Z})$. Let $A_F(n,1)$ denote the nth coefficient of the Dirichlet series for $L(s,F)$. Then for any $\alpha\in\mathbb{R}$ there exists an effective constant $c_2>0$, such that*

$$\sum_{n\leq N}\Lambda(n)A_F(n,1)e(n\alpha)\ll N\exp(-c_2\sqrt{\log N}),$$

where the implied constant depends only on the form F.

Theorem 1.1 shows that for a Hecke-Maass form F for $SL(2,\mathbb{Z})$, the sequences $\{\Lambda(n)\}$ and $\{a_F(n)e(n^2\alpha)\}$ are strong asymptotically orthogonal. Furthermore Theorem 1.2 gives that for a Hecke-Maass form F for $SL(3,\mathbb{Z})$, the sequences $\{\Lambda(n)\}$ and $\{A_F(n,1)e(n\theta)\}$ are strong asymptotically orthogonal.

We applied the theory of automorphic L-fucntions, the Vaughan identity, the estimation of exponential sum, and the strong orthogonality between Fourier coefficients and the additive characters to establish Theorems 1.1 and 1.2. In Section 2, we shall use Theorem 1.2 an example to illustrate our main arguments. In Section 3, we shall talk about some related results.

2. Main Steps in the Proof

By Dirichlet's theorem on rational approximations, for any $\alpha \in \mathbb{R}$ and any given $Q \geq 1$, there exist two integers l and q such that

$$\left|\alpha - \frac{l}{q}\right| \leq \frac{1}{qQ}, \quad 1 \leq q \leq Q, \quad (l,q) = 1. \tag{2.1}$$

Furthermore, we parameterize Q by setting

$$Q = N\exp(-C_0\sqrt{\log N}) \tag{2.2}$$

for some positive constant C_0 to be determined later. From the point of view of the circle method, one may split α into two pieces: α belongs to the major arcs when q is quite small (precisely $q \leq \exp(C_0\sqrt{\log N})$), and the minor arcs otherwise, where $\exp(C_0\sqrt{\log N}) < q \leq N\exp(-C_0\sqrt{\log N})$.

2.1. **The major arcs.**

The Godement-Jacquet L-function, defined for $\Re s > 1$ by

$$L(s,F) = \sum_{n=1}^{\infty} \frac{A_F(n,1)}{n^s}, \tag{2.3}$$

has an analytic continuation to the whole complex plane and satisfies the following functional equation

$$G_\nu(s)L(s,F) = \widetilde{G}_\nu(1-s)L(1-s,\widetilde{F}), \tag{2.4}$$

where

$$G_\nu(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-2\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s+\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s-1+\nu_1+2\nu_2}{2}\right),$$
$$\widetilde{G}_\nu(s) = \pi^{-\frac{3s}{2}}\Gamma\left(\frac{s+1-\nu_1-2\nu_2}{2}\right)\Gamma\left(\frac{s-\nu_1+\nu_2}{2}\right)\Gamma\left(\frac{s-1+2\nu_1+\nu_2}{2}\right),$$

and $\widetilde{F}$ is the dual Maass form of F.

We firstly proved the so-called Prime Number Theorem for the coefficients of $L(s,F)$ with multiplicative twists.

Lemma 2.1. *Let $N \geq 2$ and F be a Hecke-Maass form for $SL(3,\mathbb{Z})$. Let χ be any Dirichlet character modulo q. Suppose that α belongs to the major arcs. Then there exists a constant $c > 0$ such that*

$$\sum_{n\leq N} \Lambda(n)A_F(n,1)\chi(n) \ll q^{\frac{3}{2}}N\exp(-c\sqrt{\log N}), \tag{2.5}$$

where the implied constant only depends on the form F.

To this aim, we need various analytic properties of the twisted L-function $L(s, F\times\chi)$. In particular, the zero-free region of $L(s, F\times\chi)$ plays an important role in our argument, which states that there exists some absolute constant $c>0$ such that the region

$$\left\{s=\sigma+it : \sigma \geq 1-\frac{c}{\log(q(|t|+3))}\right\}$$

does not contain any zeros of $L(s, F\times\chi)$.

Based on Lemma 2.1, by standard arguments of analytic number theory, we have

Proposition 2.1. *Let $N\geq 2$ and $A_F(n,1)$ be nth coefficient of $L(s,F)$. Suppose that α belongs to the major arcs. Then there exists a constant $c>0$ such that*

$$\sum_{n\leq N}\Lambda(n)A_F(n,1)e(n\alpha) \ll N\exp(-c\sqrt{\log N}),$$

where the implied constant only depends on the form F.

2.2. **The minor arcs.** The L-function $L(s,F)$ can be also written as an Euler product

$$L(s,F)=\prod_p\prod_{1\leq j\leq 3}\left(1-\frac{\alpha_F(p,j)}{p^s}\right)^{-1},$$

where $\alpha_F(p,1), \alpha_F(p,2), \alpha_F(p,3)$ are local parameters.

By taking the logarithmic derivatives for $L(s,F)$, we have

$$-\frac{L'}{L}(s,F)=\sum_{n=1}^{\infty}\frac{\Lambda_F(n)}{n^s},$$

where

$$\Lambda_F(n)=\begin{cases}\log p\sum_{j=1}^{3}\alpha_F(p,j)^k, & \text{if } n=p^k,\\ 0, & \text{otherwise.}\end{cases}$$

We define the function $L^{-1}(s,F)$ as

$$L^{-1}(s,F)=\sum_{n=1}^{\infty}\frac{\mu_F(n)}{n^s},$$

where

$$\mu_F(n)=\begin{cases}0, & \text{if } p^4|n,\\ \prod_{l=1}^{3}\prod_{p^\ell\|n}(-1)^\ell\sum_{1\leq j_1<\cdots<j_\ell\leq 3}\alpha_F(p,j_1)\cdots\alpha_F(p,j_\ell), & \text{otherwise.}\end{cases}$$

It follows from the definition of $\Lambda_F(n)$ and the result of Kim and Sarnak [6], which states that $A_F(n,1) \ll n^{5/14} d_3(n)$, we have

$$\sum_{n\le N} \Lambda(n) A_F(n,1) e(n\alpha) = \sum_{n\le N} \Lambda_F(n) e(n\alpha) + O(N^{6/7+\varepsilon}).$$

To go further, we need an analogue of Vaughan's identity (see [10], [11])

$$\begin{aligned} -\frac{L'(s,F)}{L(s,F)} =& H(s) - L'(s,F)G(s) - L(s,F)H(s)G(s) \\ & + \Big(-\frac{L'(s,F)}{L(s,F)} - H(s) \Big) \Big(1 - L(s,F)G(s) \Big), \end{aligned} \tag{2.6}$$

where

$$H(s) = \sum_{n\le X} \Lambda_F(n) n^{-s} \quad \text{and} \quad G(s) = \sum_{n\le Y} \mu_F(n) n^{-s} \quad \text{for any } X, Y > 1.$$

This gives that

$$\sum_{n\le N} \Lambda_F(n) e(n\alpha) = S_1 + S_2 - S_3 + S_4 + O((XY)^{1+\epsilon})$$

with

$$\begin{aligned} S_1 &= \sum_{n\le X} \Lambda_F(n) e(n\alpha), \\ S_2 &= \sum_{m\le Y} \mu_F(m) \sum_{mn\le N} A_F(n,1)(\log n) e(mn\alpha), \\ S_3 &= \sum_{m\le XY} a_F(m) \sum_{mn\le N} A_F(n,1) e(mn\alpha), \\ S_4 &= \sum_{X<m<N/Y} \sum_{Y<n\le N/m} b_F(n) \Lambda_F(m) e(mn\alpha), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} a_F(m) &= \sum_{\substack{bc=m \\ b\le Y, c\le X}} \mu_F(b) \Lambda_F(c), \\ b_F(n) &= \sum_{\substack{bc=n \\ b>Y}} \mu_F(b) A_F(c,1). \end{aligned}$$

Since one can estimate S_1 trivially, it suffices to estimate Type I′ bilinear forms S_2, S_3 and Type II bilinear form S_4.

To estimate Type I′ bilinear forms S_2, S_3, we need the strong orthogonality between $A_F(n,1)$ and $e(n\alpha)$ proved by S. Miller [9].

Lemma 2.2. *Let $N \geq 2$ and $A_F(n,1)$ be the nth coefficient of $L(s,F)$. Then for any $\alpha \in \mathbb{R}$ and any $\epsilon > 0$ one has*

$$\sum_{n \leq N} A_F(n,1)e(n\alpha) \ll N^{\frac{3}{4}+\epsilon},$$

where the implied constant depends only on the form F and ϵ.

It is standard that S_4 can be written as a linear combination of $O(\log^2 N)$ terms, each of which is of the form

$$\sum_{\substack{m \sim M', n \sim N' \\ mn \sim x'}} \sum b_F(n)\Lambda_F(m)e(mn\alpha)$$

with

$$X < M' < \frac{2N}{Y}, \quad Y < N' < \frac{2N}{X}, \quad XY < x' < 2N, \quad M'N' \asymp x'.$$

Set $a(m) := \Lambda_F(m)$ and $b(n) := b_F(n)$. We still need the following standard estimate.

Lemma 2.3. *Let $M, N, x \geq 2$. Let $\{a(m) : 1 \leq m \leq M\}$ and $\{b(n) : 1 \leq n \leq N\}$ be any two complex-valued sequences. Suppose that there exists a constant $c_0 > 0$ such that*

$$\sum_{m \sim M} |a(m)|^2 \ll M \log^{c_0} M \text{ and } \sum_{n \sim N} |b(n)|^2 \ll N \log^{c_0} N.$$

Then there exists a positive constant c, depending on c_0, such that for any α satisfying

$$\sum_{\substack{m \sim M, n \sim N \\ mn \sim x}} \sum a(m)b(n)e(mn\alpha) \ll x\Big(\frac{1}{M} + \frac{1}{N} + \frac{1}{q} + \frac{1}{q}x\Big)^{\frac{1}{2}} \log^c x.$$

It is nontrivial to show that $a(m) := \Lambda_F(m)$ and $b(n) := b_F(n)$ satisfies the conditions in Lemma 2.3. We did this by the Selberg-Delange method and the analytic properties of automorphic L-functions.

Lemma 2.4. *Let F be a Hecke-Maass form for $SL(3,\mathbb{Z})$. Let $A_F(n,1)$, $\mu_F(n)$ and $\Lambda_F(n)$ be the coefficients defined as above. Then for $x \geq 2$ we have*

$$\sum_{n \leq x} d(n)|\mu_F(n)|^2 \ll x \log x,$$

$$\sum_{n \leq x} d(n)|A_F(n,1)|^2 \ll x \log x,$$

$$\sum_{n \leq x} d(n)|\Lambda_F(n)|^2 \ll x \log^3 x.$$

On collecting all results, and taking $X = Y = N^{\frac{1}{6}}$, we have

$$S_i \ll N \exp(-c\sqrt{\log N}), i = 1, 2, 3, 4.$$

Hence, we obtain

Proposition 2.2. *Let $N \geq 2$ and $A_F(n, 1)$ be the nth coefficient of $L(s, F)$. Suppose that α belongs to the minor arcs. Then there exists a constant $c > 0$ such that*

$$\sum_{n \leq N} \Lambda(n) A_F(n, 1) e(n\alpha) \ll N \exp(-c\sqrt{\log N}),$$

where the implied constant only depends on the form F.

From Propositions 2.1 and 2.2, we complete the proof of Theorem 1.2.

3. Further Discussion

Our previous arguments can not show that the sequences $\{\mu(n)\}$ and $\{A_F(n, 1)e(n\theta)\}$ are also strong asymptotically orthogonal. Recently we are able to show that the sequences $\{\mu(n)\}$ and $\{A_F(n, 1)e(n\theta)\}$ are actually strong asymptotically orthogonal as expected, based on the strategy of Iwaniec-Kowalski [5, Page 124] and Theorem 1.2.

Theorem 3.1. *Let $N \geq 2$ and $L(s, F)$ be the L-function associated to a Hecke-Maass form F for $SL(3, \mathbb{Z})$. Let $A_F(n, 1)$ denote the nth coefficient of the Dirichlet series for $L(s, F)$. Then for any $\theta \in \mathbb{R}$ there exists an effective constant $c > 0$, such that*

$$\sum_{n \leq N} \mu(n) A_F(n, 1) e(n\theta) \ll N \exp\left(-c (\log N)^{\frac{1}{3}}\right),$$

where the implied constant depends only on the form F.

In addition, it seems interesting to detect how the distribution of the zeros of L-functions affect the magnitude of $\sum_{n \leq x} \mu(n) e(n\theta)$. In the literature, many authors pursued in this direction. Under the assumption that for every Dirichlet character χ, the Dirichlet L-function $L(s, \chi)$ has no zeros in $\Re s > a$, the current best estimate is due to Baker and Harman [1], who proved

$$\sum_{n \leq x} \mu(n) e(n\theta) \ll \begin{cases} x^{a+\frac{1}{4}}, & \text{for } \frac{1}{2} \leq a \leq \frac{11}{20}, \\ x^{\max\left(\frac{a+1}{2}, \frac{4}{5}\right)}, & \text{for } \frac{11}{20} \leq a < 1. \end{cases}$$

To go further in our cases, we choose to make several necessary assumptions.

(A) Weaker Grand Riemann Hypothesis: For any primitive Dirichlet character χ, there is no zero of $L(F \times \chi, s)$ in the half plane $\sigma = \Re s > a$. Here $\frac{1}{2} \leq a < 1$.

(B) Hypothesis H: For any fixed $\nu \geq 2$,

$$\sum_{p} \frac{|a_F(p^\nu)|^2(\log p)^2}{p^\nu} < \infty, \tag{3.1}$$

where the arithmetic function $a_F(n)$ is defined by the coefficients of the logarithmic derivatives for $L(s,F)$.

(C) For any $\varepsilon > 0$, one has

$$\sum_{n\leq x} A_F(n,1,\dots,1)\, e(n^k\theta) \ll_F x^{b_m+\varepsilon} \tag{3.2}$$

uniformly in θ, where the implied constant depends only on the form F and $\frac{1}{2} \leq b_m < 1$.

Theorem 3.2. *Let $L(s,F)$ be the L-function associated to a Hecke-Maass form F for $SL(m,\mathbb{Z})$. Let $A_F(n,1,\dots,1)$ denote the nth coefficient of the Dirichlet series for $L(s,F)$. Then under the Hypothesis (A), (B) and (C), we have for any $\alpha \in \mathbb{R}$,*

$$S_{F,k}(x,\theta) = \sum_{n\leq x} \mu(n) A_F(n,1,\dots,1) e(n^k\theta) \ll x^{\rho_k+\varepsilon},$$

and

$$M_{F,k}(x,\theta) = \sum_{n\leq x} \Lambda(n) A_F(n,1,\dots,1) e(n^k\theta) \ll x^{\rho_k+\varepsilon}.$$

Here

$$\rho_1 = \begin{cases} a+\frac{1}{4}, & \text{for } \frac{1}{2} \leq a \leq \frac{7}{12}, \\ \max\left(\frac{a+1}{2}, \frac{5}{6}, \frac{1}{2}\left(1+\frac{1}{3-2b_m}\right)\right), & \text{for } \frac{7}{12} \leq a < 1. \end{cases}$$

$$\rho_k = \max\left(1-\frac{2(1-a)}{4^{k-1}+2}, 1-\frac{1}{3}\cdot\frac{1}{4^{k-1}}, 1-\frac{1}{4^{k-1}(1-b_m)+1}\right) \text{ for } k \geq 2.$$

The Weaker Grand Riemann Hypothesis gives the good bound for the logarithmic derivatives for $L(s,F)$ with twists, i.e. $\frac{L'}{L}(s, F\times\chi) \ll c^\varepsilon(|t|+1)^\varepsilon$ with $\sigma \geq a+\varepsilon$. This plays an important role in the proof of Theorem 3.2.

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